Probability (Textbook Chapters 17 and 18)

R. Sekar
Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?
Suppose you’re on a game show, and you’re given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what’s behind the doors, opens another door, say number 3, which has a goat. He says to you, “Do you want to pick door number 2?” Is it to your advantage to switch your choice of doors?

Describes a situation faced by contestants on a 70’s game show Let’s Make a Deal.
Let’s Make a Deal: Assumptions

- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors.
Let’s Make a Deal: Assumptions

- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player a second choice.
Let’s Make a Deal: Assumptions

- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player a second choice.
- If the host has a choice of which door to open, then he is equally likely to select each of them.
**Random variables** (aka “random quantities”)

- door concealing the car.
- door chosen by the player.
- door opened by the host to reveal a goat.
The Sample Space

Random variables (aka “random quantities”)
- door concealing the car.
- door chosen by the player.
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These variables take 3 possible values: A, B, and C, representing the three doors.
The Sample Space

- **Random variables** (aka “random quantities”)
  - door concealing the car.
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These variables take 3 possible values: A, B, and C, representing the three doors.

- **Outcome**: Values taken by random variables in any one experiment, e.g., (A, C, B) denotes:
  - the car is behind door A,
  - the player chooses door C,
  - the host opens door B
The Sample Space

- **Random variables** (aka “random quantities”)
  - door concealing the car.
  - door chosen by the player.
  - door opened by the host to reveal a goat.

These variables take 3 possible values: $A$, $B$, and $C$, representing the three doors.

- **Outcome**: Values taken by random variables in any one experiment, e.g., $(A, C, B)$ denotes:
  - the car is behind door $A$,
  - the player chooses door $C$,
  - the host opens door $B$

- **Sample space**: Set of all possible outcomes

\[ S = \left\{ (A, A, B), (A, A, C), (A, B, C), (A, C, B), (B, A, C), (B, B, A) \right\} \]
\[ \left\{ (B, B, C), (B, C, A), (C, A, B), (C, B, A), (C, C, A), (C, C, B) \right\} \]
Figure 17.2 The full tree diagram for the Monty Hall Problem. The second level indicates the door initially chosen by the player. The third level indicates the door revealed by Monty Hall.
Figure 17.2
The full tree diagram for the Monty Hall Problem. The second level
indicates the door initially chosen by the player. The third level indicates the door
revealed.
A *set of outcomes* is called an event. Examples:

- “prize is behind door C”
  \[\{(C, A, B), (C, B, A), (C, C, A), (C, C, B)\}\]
- “prize behind door first picked by the player”
  \[\{(A, A, B), (A, A, C), (B, B, A), (B, B, C), (C, C, A), (C, C, B)\}\]
- “player wins by switching”
  \[\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}\]
17.2. The Four Step Method

Figure 17.4: The tree diagram for the Monty Hall Problem, where the outcomes where the player wins by switching are denoted with a check mark.
Figure 17.4 The tree diagram for the Monty Hall Problem, where the outcomes where the player wins by switching are denoted with a check mark.
Assign edge probabilities

- **Car location:** A, B, C
- **Player's initial guess:** A, B, C
- **Door revealed:** B
- **Outcome:** (A, A, B), (A, A, C), (A, B, C), (A, C, B), (B, A, C), (B, B, A), (B, B, C), (B, C, A), (C, A, B), (C, B, A), (C, C, A), (C, C, B)
- **Switch wins:**
  - (A, A, B)
  - (A, A, C)
  - (A, B, C)
  - (A, C, B)
  - (B, A, C)
  - (B, B, A)
  - (B, B, C)
  - (B, C, A)
  - (C, A, B)
  - (C, B, A)
  - (C, C, A)
  - (C, C, B)
Assign edge probabilities
Compute outcome probabilities

- The tree diagram for the Monty Hall Problem where edge weights denote the probability of that branch being taken given that we are at the parent of that branch. For example, if the car is behind door $A$, then there is a $1/3$ chance that the player's initial selection is door $B$.

- Each outcome probability is simply the product of the probabilities on the path from the root to the outcome leaf.

<table>
<thead>
<tr>
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<td>$B$ 1/2</td>
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Computing Event Probabilities

- Assign edge probabilities
- Compute outcome probabilities
- Compute event probability:

![Tree Diagram]

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Computing Event Probabilities

- Assign edge probabilities
- Compute outcome probabilities
- Compute event probability: 6/9 = 2/3!

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Computing Event Probabilities

Assign edge probabilities

Compute outcome probabilities

Compute event probabilities
Strange Die Game

- A stranger challenges you to a game: whoever rolls higher will pay the other $10.
- To sweeten the deal, he says you can pick your die first.

![Dice A](image1.png) ![Dice B](image2.png) ![Dice C](image3.png)
Strange Die Game: A Vs B

Figure 17.6

The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each $\frac{1}{3}$.

17.3.1 Die A versus Die B

Step 1: Find the sample space.

The tree diagram for this scenario is shown in Figure 17.7. In particular, the sample space for this experiment are the nine pairs of values that might be rolled with Die A and Die B:

For this experiment, the sample space is a set of nine outcomes:

$$S = \{.2; 1/; .2; 5/; .2; 9/; .6; 1/; .6; 5/; .6; 9/; .7; 1/; .7; 5/; .7; 9/\}.$$

Step 2: Define events of interest.

We are interested in the event that the number on die A is greater than the number on die B. This event is a set of five outcomes:

$$f .2; 1/; .6; 1/; .6; 5/; .7; 1/; .7; 5/; g$$

These outcomes are marked A in the tree diagram in Figure 17.7.

Step 3: Determine outcome probabilities.

To find outcome probabilities, we first assign probabilities to edges in the tree diagram. Each number on each die comes up with probability $\frac{1}{3}$, regardless of the value of the other die. Therefore, we assign all edges probability $\frac{1}{3}$. The probability of an outcome is the product of the probabilities on the corresponding root-to-leaf path, which means that every outcome has probability $\frac{1}{9}$. These probabilities are recorded on the right side of the tree diagram in Figure 17.7.
Strange Die Game: A Vs C

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each $\frac{1}{3}$.

17.3.1 Die A versus Die B

Step 1: Find the sample space.

The tree diagram for this scenario is shown in Figure 17.7. In particular, the sample space for this experiment are the nine pairs of values that might be rolled with Die A and Die B:

For this experiment, the sample space is a set of nine outcomes:

$$S = \{(2, 1), (2, 5), (2, 9), (6, 1), (6, 5), (6, 9), (7, 1), (7, 5), (7, 9)\}.$$

Step 2: Define events of interest.

We are interested in the event that the number on die A is greater than the number on die B. This event is a set of five outcomes:

$$(2, 1), (6, 1), (6, 5), (7, 1), (7, 5)$$

These outcomes are marked A in the tree diagram in Figure 17.7.

Step 3: Determine outcome probabilities.

To find outcome probabilities, we first assign probabilities to edges in the tree diagram. Each number on each die comes up with probability $\frac{1}{3}$, regardless of the value of the other die. Therefore, we assign all edges probability $\frac{1}{3}$. The probability of an outcome is the product of the probabilities on the corresponding root-to-leaf path, which means that every outcome has probability $\frac{1}{9}$. These probabilities are recorded on the right side of the tree diagram in Figure 17.7.
Figure 17.6

The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each $\frac{1}{3}$.

17.3.1 Die A versus Die B

Step 1: Find the sample space.

The tree diagram for this scenario is shown in Figure 17.7. In particular, the sample space for this experiment are the nine pairs of values that might be rolled with Die A and Die B:

For this experiment, the sample space is a set of nine outcomes:

$$S = \{2; 1; 6; 1; 6; 5; 7; 1; 7; 5\}.$$ 

Step 2: Define events of interest.

We are interested in the event that the number on die A is greater than the number on die B. This event is a set of five outcomes:

$$\{2; 1; 6; 1; 6; 5; 7; 1; 7; 5\}.$$ 

These outcomes are marked A in the tree diagram in Figure 17.7.

Step 3: Determine outcome probabilities.

To find outcome probabilities, we first assign probabilities to edges in the tree diagram. Each number on each die comes up with probability $\frac{1}{3}$, regardless of the value of the other die. Therefore, we assign all edges probability $\frac{1}{3}$. The probability of an outcome is the product of the probabilities on the corresponding root-to-leaf path, which means that every outcome has probability $\frac{1}{9}$. These probabilities are recorded on the right side of the tree diagram in Figure 17.7.
Strange Die Game: Sum of Two Rolls Wins

![Die A](image1.png)
![Die B](image2.png)
![Die C](image3.png)

Figure 17.6 The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each $\frac{1}{3}$.

17.3.1 Die A versus Die B

Step 1: Find the sample space.

The tree diagram for this scenario is shown in Figure 17.7. In particular, the sample space for this experiment are the nine pairs of values that might be rolled with Die A and Die B:

$$S = \{2; 1; 2; 5; 2; 9; 6; 1; 6; 5; 6; 9; 7; 1; 7; 5; 7; 9\}$$

Step 2: Define events of interest.

We are interested in the event that the number on die A is greater than the number on die B. This event is a set of five outcomes:

$$\{2; 1; 6; 1; 6; 5; 7; 1; 7; 5\}$$

These outcomes are marked A in the tree diagram in Figure 17.7.

Step 3: Determine outcome probabilities.

To find outcome probabilities, we first assign probabilities to edges in the tree diagram. Each number on each die comes up with probability $\frac{1}{3}$, regardless of the value of the other die. Therefore, we assign all edges probability $\frac{1}{3}$. The probability of an outcome is the product of the probabilities on the corresponding root-to-leaf path, which means that every outcome has probability $\frac{1}{9}$. These probabilities are recorded on the right side of the tree diagram in Figure 17.7.
Strange Die Game: Sum of Two Rolls Wins

Figure 17.6

The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die $A$, the probabilities of getting a 2, 6, or 7 are each $\frac{1}{3}$.

17.3.1 Die $A$ versus Die $B$

Step 1: Find the sample space.

The tree diagram for this scenario is shown in Figure 17.7. In particular, the sample space for this experiment are the nine pairs of values that might be rolled with Die $A$ and Die $B$:

For this experiment, the sample space is a set of nine outcomes:

$$S = \{ (2, 1), (6, 1), (6, 5), (7, 1), (7, 5) \}.$$  

Step 2: Define events of interest.

We are interested in the event that the number on die $A$ is greater than the number on die $B$. This event is a set of five outcomes:

$$\{ (2, 1), (6, 1), (6, 5), (7, 1), (7, 5) \}.$$  

These outcomes are marked $A$ in the tree diagram in Figure 17.7.

Step 3: Determine outcome probabilities.

To find outcome probabilities, we first assign probabilities to edges in the tree diagram. Each number on each die comes up with probability $\frac{1}{3}$, regardless of the value of the other die. Therefore, we assign all edges probability $\frac{1}{3}$. The probability of an outcome is the product of the probabilities on the corresponding root-to-leaf path, which means that every outcome has probability $\frac{1}{9}$. These probabilities are recorded on the right side of the tree diagram in Figure 17.7.
Figure 17.6

The strange dice. The number of pips on each concealed face is the same as the number on the opposite face. For example, when you roll die A, the probabilities of getting a 2, 6, or 7 are each $\frac{1}{3}$.

17.3.1 Die A versus Die B

Step 1: Find the sample space.

The tree diagram for this scenario is shown in Figure 17.7. In particular, the sample space for this experiment are the nine pairs of values that might be rolled with Die A and Die B:

For this experiment, the sample space is a set of nine outcomes:

$$S = \{.(2; 1); (6; 1); (6; 5); (7; 1); (7; 5)\}.$$ 

Step 2: Define events of interest.

We are interested in the event that the number on die A is greater than the number on die B. This event is a set of five outcomes:

$$\{.(2; 1); (6; 1); (6; 5); (7; 1); (7; 5)\}.$$ 

These outcomes are marked A in the tree diagram in Figure 17.7.

Step 3: Determine outcome probabilities.

To find outcome probabilities, we first assign probabilities to edges in the tree diagram. Each number on each die comes up with probability $\frac{1}{3}$, regardless of the value of the other die. Therefore, we assign all edges probability $\frac{1}{3}$. The probability of an outcome is the product of the probabilities on the corresponding root-to-leaf path, which means that every outcome has probability $\frac{1}{9}$. These probabilities are recorded on the right side of the tree diagram in Figure 17.7.
Birthday Problem

What is the probability of finding two people with the same birthday in this class?
Birthday Problem

- What is the probability of finding two people with the same birthday in this class?

- The probability that two students have different birthdays: \( \frac{364}{365} \)

- In a class of \( n \), there are \( \binom{n}{2} \) pairs of students to consider.
  - If we assume that whether one pair shares a birthday is independent of another, we can simply multiply these probabilities

\[
Pr(\text{no two persons with same birthday}) \approx \left( \frac{364}{365} \right)^{\binom{n}{2}} \approx \left( \frac{364}{365} \right)^{n^2/2}
\]

- For \( n = 44 \), this formula yields a probability of 7%
  - \( n = 23 \) is enough to have better than even chance of finding two with the same birthday.
Birthday Problem: More Accurate Approach

- What is the probability of finding two people with the same birthday in this class?

- There are $365^n$ possible sequences of birthdays for $n$ people
  - We assume these are all equally likely

- Number of sequences without repetition: $365 \cdot 364 \cdots (365 - (n - 1))$

- Probability that no two of $n$ persons have same birthday:
  $$\frac{365}{365} \cdot \frac{365 - 1}{365} \cdots \frac{365 - (n - 1)}{365} = \left(1 - \frac{0}{365}\right) \left(1 - \frac{1}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)$$

- Use the approximation $(1 - x) < e^{-x}$ to derive an upper bound:
  $$Pr(\text{no two persons with same birthday}) < e^0 \cdot e^{\frac{-1}{365}} \cdot e^{\frac{-n-1}{365}} = e^{\sum_{i=1}^{n-1} i} = e^{\frac{-n(n-1)}{2 \cdot 365}}$$

- For $n = 44$, this evaluates to 7.5%
A countable *sample space* $S$ is a nonempty countable set.

An *outcome* $\omega$ is an element of $S$.

A *probability function* $Pr : S \rightarrow \mathbb{R}$ is a total function such that

- $Pr[\omega] \geq 0$ for all $\omega \in S$, and
- $\sum_{\omega \in S} Pr[\omega] = 1$
Set Theory and Probability

- A countable *sample space* $S$ is a nonempty countable set.
- An *outcome* $\omega$ is an element of $S$.
- A *probability function* $Pr : S \rightarrow \mathbb{R}$ is a total function such that
  - $Pr[\omega] \geq 0$ for all $\omega \in S$, and
  - $\sum_{\omega \in S} Pr[\omega] = 1$
- An *event* $E$ is a subset of $S$. Its probability is given by:
  $$Pr[E] = \sum_{\omega \in E} Pr[\omega]$$
Probability Rules from Set Theory

Many probability rules follow from the rules on set cardinality

Sum Rule: If $E_0, E_1, \ldots, E_n, \ldots$ are pairwise disjoint events, then

$$Pr[\bigcup_{n \in \mathbb{N}} E_n] = \sum_{n \in \mathbb{N}} Pr[E_n]$$
Probability Rules from Set Theory

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**Complement Rule:** \( Pr[A] = 1 - Pr[A] \)
Probability Rules from Set Theory

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**Difference Rule:**

$$Pr[B - A] = Pr[B] - Pr[A \cap B]$$
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$$\Pr[B - A] = \Pr[B] - \Pr[A \cap B]$$

**Inclusion–Exclusion:**

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

**Union Bound:** $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$
Probability Rules from Set Theory

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**Inclusion–Exclusion:**

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

**Union Bound:** $Pr[A \cup B] \leq Pr[A] + Pr[B]$

**Monotonicity:** $A \subseteq B \rightarrow Pr[A] \leq Pr[B]$
Uniform Probability Spaces

A finite probability space $S$ said to be uniform if $Pr[\omega]$ is the same for all $\omega$. In such spaces:

$$Pr[E] = \frac{|E|}{|S|}$$

We often this assumption — for instance, whenever probability was brought up while counting.
Two players take turns flipping fair coins. The first one to land heads wins. What is the probability of each player winning?
Conditional Probability

- Probability of an event under a condition
- The condition limits consideration to a subset of outcomes
  - Consider this subset (rather than whole of $S$) as the space of all possible outcomes

\[ Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]} \]
Conditional Probability

- Probability of an event under a condition
- The condition limits consideration to a subset of outcomes
  - Consider this subset (rather than whole of $S$) as the space of all possible outcomes

$$Pr[X|Y] = \frac{Pr[X \cap Y]}{Pr[Y]}$$

**Example:** $Pr[\text{win by switching} \mid \text{pick } A \text{ and goat at } B]$

$$Pr(\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\} \mid \{(A, A, B), (A, A, C), (C, A, B)\})$$

i.e., $Pr[\{(C, A, B)\}] / Pr[\{(A, A, B), (A, A, C), (C, A, B)\}]$

which evaluates to $1/2$ — switching does not seem to help!
Wrong Question: $Pr[\text{win by switching} \mid \text{pick A and goat at B}]$  

$Pr(\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\} \mid \{(A, A, B), (A, A, C), (C, A, B)\})$  

$= Pr[\{(C, A, B)\}] / Pr[\{(A, A, B), (A, A, C), (C, A, B)\}] = \frac{1/9}{1/18 + 1/18 + 1/9} = 1/2$  

Switching does not seem to help!
Wrong Question: $Pr[\text{win by switching} \mid \text{pick } A \text{ and goat at } B]$

$Pr(\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\} \mid \{(A, A, B), (A, A, C), (C, A, B)\})$

$= Pr(\{(C, A, B)\}) / Pr(\{(A, A, B), (A, A, C), (C, A, B)\}) = \frac{1/9}{1/18+1/18+1/9} = 1/2$

Switching does not seem to help!

Right Question: $Pr[\text{win by switching} \mid \text{pick } A \text{ and host opens } B]$

$Pr(\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\} \mid \{(A, A, B), (C, A, B)\})$

$= Pr(\{(C, A, B)\}) / Pr(\{(A, A, B), (C, A, B)\}) = \frac{1/9}{1/18+1/9} = 2/3$

Switching does help: The main clue is the host’s decision to open $B$!
Best-of-Three Playoff

Both teams have a 0.5 probability of winning the first match. But for subsequent games, the winning team has a 2/3 probability of winning the next match. Similarly, the losing team has a 2/3 probability of losing the next match.

*What is the probability that the team that wins the first match will win the playoffs?*
The Four-Step Method for Conditional Probability

18.3. The Four-Step Method for Conditional Probability

This is a question about a conditional probability. Let $A$ be the event that the local team wins the tournament, and let $B$ be the event that they win the first game. Our goal is then to determine the conditional probability $P(A|B)$.

We can tackle conditional probability questions just like ordinary probability problems: using a tree diagram and the four step method. A complete tree diagram is shown in Figure 18.1.

**Figure 18.1**
The tree diagram for computing the probability that the local team wins two out of three games given that they won the first game.

**Step 1: Find the Sample Space**
Each internal vertex in the tree diagram has two children, one corresponding to a win for the local team (labeled $W$) and one corresponding to a loss (labeled $L$). The complete sample space is:

$$S = \{WW, WLW, WLL, LWL, LWW, LL\}$$

**Step 2: Define Events of Interest**
The event that the local team wins the whole tournament is:

$$T = \{WW, WLW, LWL\}$$

And the event that the local team wins the first game is:

$$F = \{WW, WLW, WLL\}$$

<table>
<thead>
<tr>
<th>game 1</th>
<th>game 2</th>
<th>game 3</th>
<th>outcome</th>
<th>event A: win the series</th>
<th>event B: win game 1</th>
<th>outcome probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>$2/3$</td>
<td>$W$</td>
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<td>✓</td>
<td>$1/9$</td>
</tr>
<tr>
<td>$L$</td>
<td>$1/3$</td>
<td>$W$</td>
<td>$W$</td>
<td>✓</td>
<td>✓</td>
<td>$1/9$</td>
</tr>
<tr>
<td>$L$</td>
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<td>✓</td>
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<td>✓</td>
<td>✓</td>
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</tr>
</tbody>
</table>
Four-Step Method for Conditional Probability

- Find the sample space
  \[ S = \{ \text{WW, WLW, WLL, LWW, LWL, LL} \} \]

- Define events of interest
  \[ W_T = \{ \text{WW, WLW, LWW} \} \]
  \[ W_F = \{ \text{WW, WLW, WLL} \} \]

- Determine outcome probabilities
  - Outcomes correspond to the tree leaves, and are annotated with their probabilities

- Compute event probabilities

\[
\Pr[W_T] = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{3} + \frac{1}{18} + \frac{1}{9} = \frac{1}{2}
\]

\[
\Pr[W_T|W_F] = \frac{\Pr[\{WW, WLW\}]}{\Pr[W_F]} = \frac{1/3 + 1/18}{1/2} = \frac{7}{9}
\]
What are Edge Probabilities in Tree Diagrams?

- They are just conditional probabilities!

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<td>1/3</td>
<td>WLL</td>
<td>✓</td>
<td></td>
<td>1/9</td>
</tr>
<tr>
<td>L</td>
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<td>1/3</td>
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<td>✓</td>
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<td>1/18</td>
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<td>1/3</td>
<td>LWW</td>
<td>✓</td>
<td></td>
<td>1/9</td>
</tr>
<tr>
<td>L</td>
<td>L</td>
<td>1/3</td>
<td>LL</td>
<td>✓</td>
<td></td>
<td>1/3</td>
</tr>
</tbody>
</table>

18.3. The Four-Step Method for Conditional Probability

local team wins the tournament, given that they win the first game?

This is a question about a conditional probability. Let $A$ be the event that the local team wins the tournament, and let $B$ be the event that they win the first game.

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And the event that the local team wins the first game is:

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Extending Probability Rules for Conditional Probability

Product Rule 2: \( \Pr[E_1 \cap E_2] = \Pr[E_1] \cdot \Pr[E_2|E_1] \)
Extending Probability Rules for Conditional Probability

Product Rule 2: \( Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2|E_1] \)

Product Rule 3: \( Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2|E_1] \cdot Pr[E_3|E_1 \cap E_2] \)
Extending Probability Rules for Conditional Probability

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Bayes’ Rule: \( \Pr[B|A] = \frac{\Pr[A|B] \cdot \Pr[B]}{\Pr[A]} \)
Extending Probability Rules for Conditional Probability

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Bayes’ Rule: \( \Pr[B|A] = \frac{\Pr[A|B] \cdot \Pr[B]}{\Pr[A]} \)

Total Probability Law: \( \Pr[A] = \Pr[A|E] \cdot \Pr[E] + \Pr[A|\bar{E}] \cdot \Pr[\bar{E}] \)

Total Probability Law 2: If \( E_i \) are mutually disjoint and \( \Pr[\bigcup E_i] = 1 \) then
\( \Pr[A] = \sum \Pr[A|E_i] \cdot \Pr[E_i] \)

Inclusion-Exclusion: \( \Pr[A \cup B|C] = \Pr[A|C] + \Pr[B|C] - \Pr[A \cap B|C] \)
Independence

- An event $A$ is independent of $B$ iff the following (equivalent) conditions hold:
  - $Pr[A|B] = Pr[A]$
  - $Pr[A \cap B] = Pr[A] \cdot Pr[B]$
  - $B$ is independent of $A$

- Often, independence is an assumption.

- Definition can be generalized to 3 (or $n$) events. Events $E_1$, $E_2$ and $E_3$ are mutually independent iff all of the following hold:
  - $Pr[E_1 \cap E_2] = Pr[E_1] \cdot Pr[E_2]$
  - $Pr[E_2 \cap E_3] = Pr[E_2] \cdot Pr[E_3]$
  - $Pr[E_1 \cap E_3] = Pr[E_1] \cdot Pr[E_3]$
  - $Pr[E_1 \cap E_2 \cap E_3] = Pr[E_1] \cdot Pr[E_2] \cdot Pr[E_3]$
Medical Testing

False Positive (FP): \( Pr[\text{positive test} \mid \text{not sick}] \)

In the context of statistical hypothesis testing:

- FP is called *type I error* or *significance* and denoted by the letter \( \alpha \)
- \( \gamma = 1 - \alpha \) is called *specificity* or *confidence* of the test.

False Negative: \( Pr[\text{negative test} \mid \text{sick}] \)

In statistical hypothesis testing,

- FN is called *type II error* and denoted \( \beta \).
- \( 1 - \beta \) is called the *power* of the test.
Medical Testing

- Consider a diagnostic test with FP and FN probabilities of 0.05 and 0.02 respectively.

- If a test comes back positive, what is the likelihood that he/she has the disease?
Consider a diagnostic test with FP and FN probabilities of 0.05 and 0.02 respectively.

If a test comes back positive, what is the likelihood that he/she has the disease? It depends ...
Consider a diagnostic test with FP and FN probabilities of 0.05 and 0.02 respectively.

If a test comes back positive, what is the likelihood that he/she has the disease?
It depends ...
... on what fraction of the tested population is actually sick.
Assume this is 1%. 

Medical Testing
Medical Testing: Four-Step Method

- Find the sample space

\[ S = \{(\text{sick}, \text{pos}), (\text{sick}, \text{neg}), (\neg \text{sick}, \text{pos}), (\neg \text{sick}, \text{neg})\} \]

- Define events of interest

\[ \text{Sick} = \{(\text{sick}, \text{pos}), (\text{sick}, \text{neg})\} \quad \text{Pos} = \{(\text{sick}, \text{pos}), (\neg \text{sick}, \text{pos})\} \]

- Determine outcome probabilities: See the tree diagram on the previous slide

- Compute conditional probability

\[
\begin{align*}
\Pr[\text{Pos}] &= \Pr[(\text{sick}, \text{pos})] + \Pr[(\neg \text{sick}, \text{pos})] = 0.01 \cdot 0.98 + 0.99 \cdot 0.05 = 0.0593 \\
\Pr[\text{Sick}|\text{Pos}] &= \frac{\Pr[(\text{sick}, \text{pos})]}{\Pr[\text{Pos}]} = \frac{0.01 \cdot 0.98}{0.0593} = 16.5\%
\end{align*}
\]

Although the test is more than 95% accurate, a positive does not mean much: You have only a small (16.5%) chance of being actually sick!
Medical Testing: Summary

- While false positives are rare, they are more common than the likelihood of a random person being sick.
  - In fact, the condition being tested is 5x less prevalent than FPs.
  - So, 4 out 5 times, people flagged by the test are not sick.

- This calculation is based on the assumption that the person being tested is someone picked randomly from the population.
  - If we tested only those that display symptoms of the sickness, the rates will be different.
    - In particular, we need to use the prevalence of sickness among such symptomatic people.